

Transition from Abelian to non-Abelian FQHE states

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We study the transition from the Abelian multi-component (3, 3, 1) quantum Hall state to the non-Abelian one component Pfaffian state in bilayer two dimensional electron systems. We show that tunneling between layers can induce this transition. At the transition points part of the degrees of freedom that describe the (3, 3, 1) state disappear from the spectrum, and the system is correctly described by the Pfaffian state, with quasi-particles that satisfy non-Abelian statistics. The mechanism described in this work provides for a physical Hamiltonian interpretation of the algebraic projection from the (3, 3, 1) to the Pfaffian state that has been discussed in the literature.

Even denominator states in double layer two dimensional electron systems (2DES) have been observed experimentally [1] and are theoretically quite well understood [2]. The two 2DES are separated by a potential barrier that, if high and thick enough, will inhibit both Coulomb interactions and tunneling between layers. If the barrier is made thinner, Coulomb interactions will become important even if tunneling is still suppressed. The relevant parameter to measure this effect is the ratio d/l_0 where d is the interlayer separation and l_0 is the magnetic length. In real samples neither Coulomb interactions nor tunneling can be completely neglected. Therefore a very rich phase diagram can be constructed with Coulomb interlayer interaction (or alternatively the distance d) on one axis and the tunneling amplitude on the other.

We will concentrate here on systems in which a quantized Hall plateau exists at total filling fraction $\nu = 1/2$. The phase diagram for these systems was first discussed

by Halperin [3]. He assumed that the actual spin of the electrons was polarized in the direction of the external field, and that the two layers were completely equivalent. A possible experimental realization for this system is a single wide quantum well in which the self consistent Coulomb potential creates a barrier in the middle of the well with maxima in the electron density at the two edges. Halperin suggested [3] that for an intermediate range of distances d and vanishing tunneling, the so called (3, 3, 1) state should be a stable phase for the system.

The (3, 3, 1) state is a correlated bilayer state which is basically stabilized by Coulomb interactions. It was shown [4] that if the layer separation is large enough the state collapses into decoupled layers due to the fact that interlayer Coulomb interactions become negligible. The variational wave function describing this state, proposed by Halperin [5] in the context of spinful systems has the form

$$\Psi_{331} = \prod_{i<j} (z_i - z_j)^3 \prod_{i<j} (w_i - w_j)^3 \prod_{i,j} (z_i - w_j) e^{-\frac{1}{4} \sum (|z_i|^2 + |w_i|^2)}, \quad (1)$$

where z_i and w_i are the coordinates of the electrons in each plane. The first two factors represent the correlations within each layer, and the last one corresponds to the intralayer correlations. The $\nu = 1/2$ state was observed experimentally and it was checked numerically that its properties are indeed well described by the (3, 3, 1) wave function (1) [2].

It was also conjectured [3] that a transition to a Pfaffian state should occur, within the range of distances d for which the (3, 3, 1) state is stable at vanishing tunneling, when the tunneling amplitude is made large enough. The Pfaffian state, proposed by Moore and Read [6], is a candidate for a fractional quantum Hall state at $\nu = 1/2$ in single layer systems, or in general for $\nu = 1/q$ where q is an even number (were we working with bosons with strong repulsive interactions, q would be an odd integer [7]). Its variational wave function is given by

$$\Psi_{Pf} = \text{Pfaff} \left(\frac{1}{z_i - z_j} \right) \prod_{i<j} (z_i - z_j)^q e^{-\frac{1}{4} \sum |z|^2}, \quad (2)$$

where the Pfaffian is defined for a $2N \times 2N$ antisymmetric matrix whose elements are M_{ij} by

$$\text{Pfaff}(M_{ij}) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \text{sgn}(\sigma) \prod_{k=1}^N M_{\sigma(2k-1), \sigma(2k)} \quad (3)$$

or as the square root of the determinant of M . It was shown [6] that this wave function arises from applying Wick's theorem to real fermion fields, or as the real space BCS wave function for pairing of spinless fermions.

The $\nu = 1/2$ states were extensively studied in experiments [8,9] in a wide single quantum well sample, varying the well width and sheet density. It was concluded

that the state observed was the $(3, 3, 1)$, *i.e.* the Pfaffian state did not show up within the range of tunneling amplitude and thickness scanned in the experiments. The authors argued nevertheless that it should still appear in the phase diagram for larger tunneling.

Our goal is to explore the above mentioned transition between the $(3, 3, 1)$ and the Pfaffian states. Therefore, we consider a system in which the interlayer separation d is kept fixed, while the tunneling amplitude between layers can be changed arbitrarily. In other words, we will be looking at Halperin's phase diagram for a given value of the interlayer separation, such that if the tunneling amplitude vanishes, the $(3, 3, 1)$ state is the stable phase of the system.

We start with the usual chiral boson approach for the edge theory of the $(3, 3, 1)$ state (see *e.g.* [10]), that was recently reviewed in [11] with the inclusion of tunneling between layers. We further include a chemical potential term for the electrons. In this case the original theory, written in terms of two chiral bosons (a $c = 2$ central charge Conformal Field Theory (CFT)), can be mapped into an effective theory with one chiral boson and two Majorana fermions. We then study the phase diagram as a function of electron tunneling λ and chemical potential μ .

As we have already mentioned, given that the spacing between layers is kept fixed at a value such that both phases are stable, the Pfaffian state could describe a double layer sample in the limit in which the tunneling amplitude between the layers is large enough so as the two species of electrons of the $(3, 3, 1)$ state become indistinguishable. Since the edge theory for the Pfaffian state can be described by a $c = 3/2$ CFT [6], the question is then how does this process occur physically, *i.e.* how does the $(3, 3, 1)$ CFT with $c = 2$ evolve to the Pfaffian CFT with $c = 3/2$. Related to this, it has been shown [7,14,15] that the $(3, 3, 1)$ edge theory can be seen as the enveloping theory for the non Abelian Pfaffian state. Indeed, there is an algebraic procedure by which the two elementary quasi-holes of the $(3, 3, 1)$ state merge into one in the Pfaffian state by getting rid of an Ising CFT factor from the original edge theory. However, an explicit mechanism implementing physically this procedure is, to our knowledge, still lacking. In this letter we address this issue and show that electron tunneling between layers is capable of implementing this projection. More precisely, when the tunneling amplitude and/or the chemical potential increase, there is a critical line in the (λ, μ) plane at which one of the degrees of freedom that describes the original theory disappears. We furthermore show that the remaining degrees of freedom acquire the quantum numbers of the elementary excitations for the Pfaffian state and non-Abelian statistics emerges.

The edge theory for the $(3, 3, 1)$ state is described by the Hamiltonian [10]

$$H = \frac{1}{4\pi} \int dx V_{ij} : \partial_x u_i \partial_x u_j : , \quad (4)$$

where colons denote standard normal ordering. Here x is the coordinate along the edge, the u_i are chiral bosonic fields whose compactification radius is 1, and V_{ij} is a symmetric matrix whose coefficients depend on the confining potential and the interparticle interactions at the edge,

$$V = \begin{pmatrix} v & g \\ g & v \end{pmatrix} . \quad (5)$$

The commutation relations for the bosonic fields are

$$[u_i(x, t), u_j(x', t)] = i\pi K_{ij} \text{sgn}(x - x') , \quad (6)$$

where K is a symmetric matrix which characterizes the topological properties of the system

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} . \quad (7)$$

There exists an orthogonal transformation that diagonalizes V and K simultaneously, after which the Hamiltonian eq. (4) simply reads

$$H = \frac{1}{4\pi} \int dx [v_c : (\partial_x \phi_c)^2 : + v_n : (\partial_x \phi_n)^2 :] , \quad (8)$$

where $v_c = 4(v + g)$ and $v_n = 2(v - g)$. Notice that the condition $\det V > 0$ must hold in order that both modes have the same chirality. ϕ_c and ϕ_n refer to charged and neutral modes respectively, which are chiral bosons with standard commutation relations

$$[\phi_i(x, t), \phi_j(x', t)] = i\pi \delta_{ij} \text{sgn}(x - x') . \quad (9)$$

The electron operators can be written in this basis as follows

$$\begin{aligned} \psi_{e1} &\propto : e^{i(-\sqrt{2}\phi_c + \phi_n)} : \\ \psi_{e2} &\propto : e^{i(-\sqrt{2}\phi_c - \phi_n)} : \end{aligned} \quad (10)$$

while the quasi-particle operators are

$$\begin{aligned} \psi_{qp1} &\propto : e^{-i(\frac{1}{\sqrt{8}}\phi_c + \frac{1}{2}\phi_n)} : \\ \psi_{qp2} &\propto : e^{-i(\frac{1}{\sqrt{8}}\phi_c - \frac{1}{2}\phi_n)} : . \end{aligned} \quad (11)$$

In reference [11] the authors considered the problem of adding uniform electron tunneling to the edge theory. Here we will study the same problem adding also a chemical potential for the electrons. Therefore we add to the Hamiltonian the following perturbation terms

$$\begin{aligned} H' &= -\mu_0 \int dx [: \psi_{e1}^\dagger \psi_{e1} + \psi_{e2}^\dagger \psi_{e2} :] \\ &\quad + \lambda_0 \int dx [: \psi_{e1}^\dagger \psi_{e2} + \psi_{e2}^\dagger \psi_{e1} :] . \end{aligned} \quad (12)$$

Using the bosonic representation for the electron operators we can write

$$\begin{aligned}
& : \psi_{e1}^\dagger \psi_{e1} + \psi_{e2}^\dagger \psi_{e2} : \propto (i2\sqrt{2}a_0 \partial_x \phi_c - a_0^2 : (\partial_x \phi_n)^2 :) \\
& : \psi_{e1}^\dagger \psi_{e2} + \psi_{e2}^\dagger \psi_{e1} : \propto : e^{-i2\phi_n} + e^{i2\phi_n} : ,
\end{aligned} \tag{13}$$

where a_0 is the UV cut-off. In terms of these bosons the total Hamiltonian can be decoupled into charged (H_c) and neutral (H_n) sectors given by

$$\begin{aligned}
H_c &= \int dx \left[\frac{1}{4\pi} v_c : (\partial_x \phi_c)^2 : - \mu_c : \partial_x \phi_c : \right] \\
H_n &= \int dx \frac{1}{4\pi} (v_n + \mu_n) : (\partial_x \phi_n)^2 : \\
&\quad - \int dx \lambda : (e^{-i2\phi_n} + e^{i2\phi_n}) : ,
\end{aligned}$$

where $\mu_c, \mu_n \propto \mu_0$ and $\lambda \propto \lambda_0$.

The properties of the charged sector are not changed by the perturbation since the new term is linear in derivatives and can be absorbed by a shift in the bare Hamiltonian.

As for the neutral mode, it proves useful to decompose it (through conformal embedding) in terms of two chiral Majorana fermions [11]

$$: e^{-i\phi_n} : \propto (\chi_1 + i\chi_2) . \tag{15}$$

The Hamiltonian then reads

$$H_n = -\frac{1}{2} \int dx (i(v_n + \mu_n + \lambda_{eff}) : \chi_1 \partial_x \chi_1 : + i(v_n + \mu_n - \lambda_{eff}) : \chi_2 \partial_x \chi_2 :) , \tag{16}$$

where $\lambda_{eff} \propto \lambda$ [12].

We see that the two chiral Majorana fermions behave as free fields, but acquire different velocities which are determined by the bare velocity of the neutral boson v_n , the tunneling amplitude λ_{eff} and the chemical potential μ_n . Moreover, each Majorana sector describes a (chiral) Ising CFT.

It is clear now that, assuming that the perturbative treatment of the interaction Hamiltonian (12) is valid, there are two lines in the (λ_{eff}, μ_n) plane, given by $\mu_n = -(v_n \pm \lambda_{eff})$, on which one of the Majorana velocities vanishes. Though this observation is immediate from eq. (16), the study of the emerging state is non-trivial and constitutes the main result in the present work. The key observation is that when one of these two conditions is satisfied, the corresponding Ising sector disappears from the spectrum and, as we shall see, the remaining degrees of freedom describe the physics of the Pfaffian state. In fact, the Hamiltonian density for the zero-velocity mode vanishes, therefore its energy-momentum tensor and hence its central charge vanish. In this way, the central charge of the original system (the $(3, 3, 1)$ state) decreases by $1/2$. The remaining system is described by one chiral boson and one Majorana fermion with total central charge $c_{eff} = 3/2$, which is the correct value for describing the Pfaffian state.

To make sure that the projection procedure drives the system to the Pfaffian state, we now show how the electron and quasi-particle operators (10), (11) in the $(3, 3, 1)$ phase come to describe the corresponding operators in this new phase. To this end, we rewrite the original electron and quasi-particle operators in terms of the charged boson and the Ising primary fields (the Majorana fermions χ_a , the spin (order) operators σ_a and their duals (disorder) μ_a , where $a = 1, 2$ labels the two Ising sectors). Therefore the electron operators for the $(3, 3, 1)$ phase in eq. (10) can be written as

$$\begin{aligned}
\psi_{e1} &\propto : e^{-i\sqrt{2}\phi_c} (\chi_1 + i\chi_2) : \\
\psi_{e2} &\propto : e^{-i\sqrt{2}\phi_c} (\chi_1 - i\chi_2) : .
\end{aligned} \tag{17}$$

The neutral components of the quasi-particle operators can be combined and represented in terms of the order and disorder fields σ_a and μ_a as

$$\begin{aligned}
& : e^{i\phi_n/2} + e^{-i\phi_n/2} : \propto \sigma_1 \otimes \sigma_2 , \\
& : e^{i\phi_n/2} - e^{-i\phi_n/2} : \propto \mu_1 \otimes \mu_2 .
\end{aligned} \tag{18}$$

This identification has been proven in reference [13] by a careful analysis of operator product expansions on both sides. Then the quasi-particle operators can be written as

$$\begin{aligned}
\psi_{qp1} + \psi_{qp2} &\propto : e^{-i\frac{1}{\sqrt{8}}\phi_c} \sigma_1 \otimes \sigma_2 : , \\
\psi_{qp1} - \psi_{qp2} &\propto : e^{-i\frac{1}{\sqrt{8}}\phi_c} \mu_1 \otimes \mu_2 : .
\end{aligned} \tag{19}$$

The vanishing of the energy-momentum tensor for one of the two Ising sectors at the critical line implements a coset construction. The essence of the coset is to project a sector out from the physical Hilbert space. In the case at hand the projected subspace corresponds to one of the Ising sectors (say $a = 2$) of the $(3, 3, 1)$ theory [7].

It should be stressed at this point that the coset projection appears in a natural way within this context. Previous treatments advocating the coset mechanism for projecting out an Ising sector were performed without any connection to a Hamiltonian description.

An important question that remains to be answered is how this projection acts on the quasi-particle and electron operators. This projection can be seen as if all primaries in the projected sector become trivial (they have vanishing conformal weights). More precisely, the quasi-particle operators (11) degenerate into a single quasi-particle operator

$$\left. \begin{aligned} \psi_{qp1} &\propto : e^{-i\frac{1}{\sqrt{8}}\phi_c} (\sigma_1 \otimes \sigma_2 + \mu_1 \otimes \mu_2) : \\ \psi_{qp2} &\propto : e^{-i\frac{1}{\sqrt{8}}\phi_c} (\sigma_1 \otimes \sigma_2 - \mu_1 \otimes \mu_2) : \end{aligned} \right\} \rightarrow \psi_{qp}^{\text{Pfaff}} \propto : e^{-i\frac{1}{\sqrt{8}}\phi_c} \sigma_1 : \quad (20)$$

(there are indeed two possible dual descriptions in terms of σ or its dual μ) describing quasi-particle excitations over the Pfaffian ground state. They have charge $e/4$

and, more importantly, exhibit non-Abelian statistics.

Besides, the two original electron operators are projected onto

$$\left. \begin{aligned} \psi_{e1} &\propto : e^{-i\sqrt{2}\phi_c} : (\chi_1 \otimes 1_2 + i1_1 \otimes \chi_2) \\ \psi_{e2} &\propto : e^{-i\sqrt{2}\phi_c} : (\chi_1 \otimes 1_2 - i1_1 \otimes \chi_2) \end{aligned} \right\} \rightarrow \psi_e^{\text{Pfaff}} \pm : e^{-i\sqrt{2}\phi_c} : \propto : e^{-i\sqrt{2}\phi_c} : \chi_{1\pm} : e^{-i\sqrt{2}\phi_c} : , \quad (21)$$

that is the Pfaffian electron operator plus a four quasi-particle bound state : $e^{-i\sqrt{2}\phi_c} : (c.f. \text{ eq. (20)})$.

Once the electron and quasi-particle operators at the edge are known, the (bulk) wave functions for both the ground state and excited states can be constructed following [6], by computing suitable correlation functions of those operators. In this way one recovers the expression in eq. (2) for the Pfaffian ground state. The computation of the wave function for four quasi-holes over the ground state shows that the quasi-particle statistics is non-Abelian.

It is worth mentioning that non-Abelian statistics arises in this context due to the fact that one of the order-disorder fields becomes trivial. The corresponding computation with the full (non projected) quasi-particle operators gives the correct Abelian statistics in the $(3, 3, 1)$ phase.

In summary, we have shown that if we add to the edge theory for the $(3, 3, 1)$ state tunneling and chemical potential terms, there exists a critical line where part of the degrees of freedom that describe the system becomes unphysical and disappears from the spectrum. This is precisely the line where the characteristic properties of the $(3, 3, 1)$ state are lost. In turn, at these points of the parameter space, the electron and quasi-particle operators of the $(3, 3, 1)$ state can be mapped into the corresponding operators of the Pfaffian state, and the statistics of quasi-particles becomes non-Abelian. The mechanism described in this work provides for a physical interpretation of the algebraic projection from the $(3, 3, 1)$ to the Pfaffian state that has been discussed in the literature [7,14,15]. The question that remains to be answered is whether the Pfaffian state corresponds to a stable phase, *i.e.* if the system remains in this state beyond the critical lines. An alternative treatment to the one presented here is eventually needed to resolve this issue within the framework of the edge theories. According to the phase diagram found by comparing the Pfaffian state bulk wave function with the real space BCS wave function for pairing of spinless fermions [16,17], the system should be in

a Pfaffian phase somewhere beyond those lines.

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